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Generalized Goertzel algorithm for computing the natural frequencies of cantilever beams

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Abstract

The paper proposes a fast algorithm for accurate estimation of frequency within a specified, narrow range. The algorithm is useful for the identification of the natural frequencies of cantilever beams for damage detections purposes. The procedure is based on the generalized Goertzel algorithm combined with apriori knowledge of the natural frequencies intervals for cantilever beams given their physical characteristics. We compare our approach with the Chirp Z-transform and several frequency or time-frequency methods to illustrate its advantages for online damage detection.

Keywords: natural frequencies, generalized Goertzel algorithm, FFT, time-frequency, spectrum, Chirp Z-transform, CZT, damage detection

1. Introduction

The detection of damages in cantilever beams using pre-computed natural frequencies is a challenging task approached by several authors using different techniques [1, 2, 3, 4]. The main non-invasive techniques for computing the natural frequencies closely related to our approach are the spectral methods like the Goertzel algorithm [5, 6], Chirp-Z transform [7, 8] or time-frequency transforms like the short-time Fourier transform (STFT) [9], wavelets [4] etc.

The major drawback of such procedures is the fact that only tiny changes in the natural frequencies occur with the introduction of a damage, and also the technical difficulties encountered when computing these frequencies. Also for online (real-time) processing, the use of fast algorithms like the FFT and if possible even faster is almost compulsory.

With the recent introduction of the generalized Goertzel algorithm in [10], it became possible to exploit this algorithm in the search for the maximum peak in the intervals where the natural frequencies of the cantilever beams could be found.

The paper is structured as follows: In section 2, we present the characteristics of the experiment i.e. the cantilever beam and the signal analyzed, supporting the practical motivation of our study. In section 3, we overview the generalized Goertzel algorithm used for natural frequencies detection and compare it with Chirp-Z transform (CZT), its natural competitor. In section 4, the proposed method for determining a set of ten natural frequencies is described, while in section 5 we compare results of our method with time-frequency methods and we present in detail the advantages with respect to each of them. Lastly, the conclusions are drawn.

1.1. Notation

In the following text we assume a discrete signal x of length N, whose samples can be complex, $\{x[n]\} = \{x[0], x[1], \ldots, x[N-1]\}$. Symbol k can represent the number (index) of the harmonic component in the DFT, thus $k \in \mathbb{N}$ as usual, however, in the framework of generalized Goertzel algorithm we allow working with $k \in \mathbb{R}$.

2. Natural frequencies of cantilever beams and identification of damages

Vibrations of the cantilever beam following a mechanical excitation are of (damped) harmonic type and therefore it is natural to try to accurately identify the frequencies contained in the oscillating waves.

A typical such signal (one of the signals we used for testing) in time domain is plotted in Fig. 1. For acquiring the signal, we placed an accelerometer on the free end of the unloaded beam. The sampling frequency was 26,500 Hz. The informative time-frequency contents of this signal is plotted in Fig. 2.

To develop an algorithm for detection and localization of damages it is necessary to have quantifiable indicators which characterize the dynamic behavior of the beam in the undamaged and the damaged state, respectively. One of the most used indicators in the non-invasive damage detection is the change in natural frequency occurring with the alteration of the beam geometrical and mechanical characteristics.

For our measurements and tests we used a steel cantilever beam having the following geometrical characteristics: length $l = 1000$ mm, width $b = 50$ mm, height $h = 5$ mm and consequently, for the undamaged state the cross-section $A = 250 \cdot 10^{-6}$ m^2, moment of inertia $I = 520.833 \cdot 10^{-12}$ m^4. The mechanical characteristics of the beam are mass density $\rho = 7850$ kg/m^3, Young's modulus $E = 2.0 \cdot 10^{11}$ N/m^2 and Poisson's ratio $\mu = 0.3$.

The damage in our case was simulated using the Finite Element Method (FEM) in 1000 points on the beam. Also real damages for direct measurements purposes were created on the beam in several most exposed points. The problem to highlight the appearance of a damage in beams using the natural frequency of a beam depends on the forces acting on it, as well as on the dimension of the damage (the cross-section reduction).

The described beam is considered as a reference. For beams with other dimensions (l, b, h) or mechanical characteristics (ρ, E, μ) the problem can be solved in a similar way by considering the scale influence.

During the numerical experiments, we aim at computing the first ten natural frequencies since this set is usually enough for the requirements of damage detection procedures like [1].

3. Goertzel algorithm and Chirp-Z Transform

From the problem description given above, it is clear that detecting tiny changes in the natural frequencies requires a procedure able to compute the signal's frequency spectrum with very fine resolution. This section is devoted to two algorithms, candidates for this purpose. The algorithms are described and compared here and a semiconclusion is drawn. All comparisons are done with the assumption that everyting what allows precomputation is precomputed.

The FFT in its basic form is not included in this selection since it computes only DFT bins which are too coarse to be useful in our problem, and the number of samples which would increase the resolution is limited by design. Moreover, as we are only interested in a pre-determined set of spectral intervals of interest, computing the complete spectrum via FFT would be a waste of resources.

3.1. Generalized Goertzel Algorithm

The original algorithm invented by G. Goertzel [5] serves to compute the k-th single DFT component of the signal $\{x[n]\}$ of length N, i.e.

$$X[k] = \sum_{n=0}^{N-1} x[n]\, \mathrm{e}^{-\mathrm{j}2\pi k \frac{n}{N}}, \quad k = 0, \ldots, N-1. \tag{1}$$

Multiplying the right side of this equation by $1 = \mathrm{e}^{\mathrm{j}2\pi k \frac{N}{N}}$ and rearranging terms leads to its equivalent,

$$X[k] = \sum_{n=0}^{N-1} x[n]\, \mathrm{e}^{-\mathrm{j}2\pi k \frac{n-N}{N}}, \tag{2}$$

which could be regarded as a convolution. Standard signal processing literature then helps to rewrite this as an IIR (infinite impulse response) linear time-invariant system. The desired spectral sample $X[k]$ is then found as the output of such system at time n:

$$y_k[n] = s[n] - \mathrm{e}^{-\mathrm{j}\frac{2\pi k}{N}} s[n-1], \tag{3}$$

where s denotes the state variable of the (second-order) IIR system.

It should be emphasized that the transition from (1) to (2) holds for integer-valued k only; in the case of $k \in \mathbb{R}$, these two formulas are generally no longer in agreement. (The period of the transformation kernel no longer corresponds to N.) In fact, when k is not integer-valued, we can no longer speak of the DFT (1), rather of the Discrete-time Fourier transform (DTFT), which is defined by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, \mathrm{e}^{-\mathrm{j}\omega n}, \ \omega \in \mathbb{R}. \tag{4}$$

With the notation $\omega_k = 2\pi \frac{k}{N}$, $k \in \mathbb{R}$, we can write that

$$X(\omega_k) = \sum_{n=0}^{N-1} x[n]\, \mathrm{e}^{-\mathrm{j}2\pi k \frac{n}{N}}, \tag{5}$$

where we exploited the compactness of the support of the signal $\{x[n]\}$.

In order to obtain the generalized Goertzel algorithm applicable for $k \in \mathbb{R}$, we extend formula (5) by unity in the form of

$$\mathrm{e}^{\mathrm{j}2\pi k\frac{N}{N}} \cdot \mathrm{e}^{-\mathrm{j}2\pi k\frac{N}{N}} = 1 \quad \text{for } k \in \mathbb{R}, \tag{6}$$

leading to

$$X(\omega_k) = \mathrm{e}^{-\mathrm{j}2\pi k} \sum_{n=0}^{N-1} x[n]\, \mathrm{e}^{\mathrm{j}2\pi k\frac{N-n}{N}}. \tag{7}$$

Now with the same techniques as in the standard case we arrive at the DTFT coefficient in the form of

$$y_k[n] = \left(s[n] - \mathrm{e}^{-\mathrm{j}\frac{2\pi k}{N}} s[n-1]\right) \cdot \mathrm{e}^{-\mathrm{j}2\pi k}. \tag{8}$$

Comparing this with the above, we indeed see that it is a generalization, since the constant $\mathrm{e}^{-\mathrm{j}2\pi k}$ equals to one for $k \in \mathbb{Z}$. In fact, the only variation compared to the standard Goertzel algorithm is the multiplication by this constant at the very end of the algorithm. Clearly, the constant $\mathrm{e}^{-\mathrm{j}2\pi k}$ affects only the phase of the result.

Article [10] deals with the derivation and properties in detail and among other things it shows that the algorithm (in fact, both the standard and the generalized) can be shortened by a few computations. The shortened generalized algorithm is summarized in Fig. 3.

Since the fast variant of the Chirp-Z transform described in section 3.2 utilizes the FFT, we now make a few remarks:

- While the signal length N is usually pushed to be a power of two for maximum FFT performance, the complexity of the Goertzel algorithm grows linearly and regularly with the length.

- The Goertzel algorithm is able to do computations on the run, i.e. each time a new sample is acquired.
- The Goertzel algorithm allows a direct determination of the frequencies from the input signal without any reordering of input or output data and thus it is suitable for online processing.
- Since Goertzel algorithm is implemented as an IIR filter, thus large N can cause propagation of the quantization error.

In [10] it was shown that for real input data of length N, the computation of the Goertzel algorithm requires $3N$ operations per single frequency, so it is more efficient than the FFT as long as the number of desired frequencies K does not exceed $2\log_2 N$. Having a window of length $N = 1024$ would therefore mean that the Goertzel algorithm is faster up to $K = 20$ frequencies.

At specific situations, even more efficient scheme can be utilized, based of combination of the FFT and Goertzel [11], however in our problem we do not assume such situation can happen in general.

3.2. Chirp-Z Transform

The Chirp-Z transform (CZT), described well in a number of sources [7, 8], is a procedure to compute a limited range of spectral frequencies, which are linearly spread over a particular range. Formally, we are interested in K spectral samples $\omega_k = \omega_0 + k\Delta\omega$, $k = 0, \dots, K-1$, i.e.

$$X(\omega_k) = \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}(\omega_0+k\Delta\omega)n}. \qquad (9)$$

Substituting $W = \mathrm{e}^{-\mathrm{j}\Delta\omega}$ and using original trick of Bluestein [12] yields

$$X(\omega_k) = W^{\frac{k^2}{2}} \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{j}\omega_0 n} W^{\frac{n^2}{2}} W^{-\frac{(k-n)^2}{2}} \tag{10}$$

for $k = 0, \ldots, K-1$, which can be treated as convolution of two sequences, namely $\{x[n]\mathrm{e}^{-\mathrm{j}\omega_0 n} W^{\frac{n^2}{2}}\}$ and $\{W^{-\frac{n^2}{2}}\}$, followed by multiplication by $W^{\frac{k^2}{2}}$. (The name of the transform comes from the fact that the signal $\{W^{-\frac{n^2}{2}}\}$ is usually called a linear chirp.) Padding of these sequences to a proper length (power of two in most cases) allows computation of (10) via fast convolution, i.e. performing the FFT on both sequences, multiplying them in the spectral domain and the reverting by the inverse FFT.

Should (9) be evaluated for K frequencies in the direct form, it would cost $8KN$ operations. The cost of the Chirp-Z transform in its "fast" version is enumerated in [13] as $20N\log_2 N + 44N$ for $K = N$ (i.e. the number of frequency points is the same as the number of time-domain samples) and this count increases when N is not a power of two. Therefore, it is more efficient to use the direct form if it holds

$$K \leq \frac{20\log_2 N + 44}{8}, \tag{11}$$

so for example if $N = 512$ the breaking point is $K = 28$ and if $N = 1024$ the breaking point is $K = 30$.

The the complexity when K is only a portion of N can be pushed lower. Actually, using a decimation scheme we arrive at $20N\log_2 K + 45N - K$ operations. This only holds in case of N being divisible by K, and being a power of two; and is somewhat higher if this is not fulfilled. (The complexity of the radix-2 FFT is assumed to be $5N\log_2 N$ in these enumerations.)

3.3. *Goertzel algorithm vs. CZT: comparing complexities*

We see that applying the Goertzel algorithm K times, starting at ω_0 and continuing by $\Delta\omega$ steps, is equivalent in terms of results to performing the CZT once with identical parameters. Thus, what counts here is the computational burden implied by these two competitors.

Direct evaluation of CZT entry-by-entry is not advantageous since $8KN > 3KN$.

Comparing the fast CZT versus Goertzel complexities, the Goertzel algorithm is preferred if $3KN < 20N\log_2 K + 45N - K$. When we neglect K at the right side which is usually small relative to N, such a situation appears if

$$K < \frac{20}{3}\log_2 K + 15. \tag{12}$$

The breaking point here is approximately $K = 53$, regardless of length N!

Since in our case the intervals of interest are known with quite high precision, we do not need more than 53 frequency points and therefore it is more advantageous to choose the Goertzel algorithm.

3.4. *Remarks*

As Figure 1 shows, a typical beam oscillatory signal decays exponentially in time. Therefore, one could suggest to involve the damping directly in the frequency analysis. In general, such situations would benefit from CZT which can be defined for exponentially decaying signals [8]. In our case, however, we analyze short time segments (1024 samples lasts for 38 ms), and during such short periods the attenuation can be neglected. Moreover, the damping can be intercorporated into the Goertzel algorithm easily and wit no additional cost, so in this point of view, the analysis in section 3.3 remains correct.

It also should be noted that the general definition of CZT no longer assumes the complex kernel to lie on the unit circle as we have it in (9). The form of CZT restricted to the unit circle is sometimes (but improperly) referred to as the Fractional Fourier Transform (FrFT) [13].

4. Proposed method

In order to apply the Goertzel algorithm to compute the natural frequencies, we first have to determine the intervals of interest for these frequencies. Recall that the energy of the signal is concentrated in narrow spectral intervals, see Fig. 2.

Using the equation of motion for a prismatic beam [14], the first ten natural frequencies of the undamaged beam were analytically calculated and they are given in the first column in Table 1. Afterwards, we have performed FEM (Finite Element Method) simulations for 1000 positions of the damage on the analyzed beam in order to determine the frequency intervals of interest, and the results are included in Table 1. The third verification step consists in direct measurements on the damaged beam on the locations where the amplitude for the ten modes have maximum values or are null or where we encountered a point of inflection. These results are also joined in the table. Using this information, we have taken an extra-safe margin of 5% on both sides (low and up) in order to create the frequency intervals which will be used further.

After this we focus on each particular interval of interest and we compute several values lying within such interval using Goertzel algorithm. If we consider, for example, the interval of interest for the sixth sinusoid, i.e. 306 to 385 Hz obtained from columns two and three of Table 1 after applying the 5% margin, we have plotted in Fig. 5 the Goertzel values in this region of interest over the time. The FFT moduli are also present in this picture to illustrate its inefficiency in finding the position of the main lobe accurately.

As the last step a peak-picking technique is applied in order to find the frequency with the maximum amplitude. Actually, the technique is very simple — we take the maximum value independently in each time segment. In Fig. 6, we present a plot of the progress in the peak-frequency values over time using mean and median filters, respectively.

The resultant precise single frequency value per interval is computed by taking median from the median series. The reason is that the median is more "robust" to outliers which can possibly appear, especially when signal to noise ratio is adverse.

The algorithm takes as input the vibration measurements of the analyzed beam and it determines with priority the frequency intervals where the natural frequencies would lie. Since we have performed already FEM simulation in 1000 possible damaged points, and this results are stored in a database, the interest intervals are directly identified. In order to fulfill the requirement of the Goertzel algorithm we need to determine the sampling interval and the window size to match the range of values. With all these prerequisites, the Goertzel computation could take place in order to automatically obtain the set of ten natural frequencies.

The steps of the algorithm are summarized in Fig. 4.

5. Comparison with other identification methods

This section is devoted to comparing properties of different methods used for identification of natural frequencies in cantilever beams. In paper [9], a systematic overview of the time-frequency methods used to extract the natural frequency of cantilever beams was presented. The time-frequency transforms used in there to extract the natural frequencies were the short-time Fourier transform, the Wigner transform and the non-stationary Gabor transform. Furthermore, the spectra of these time-frequency transforms was post-processed before the extraction of the frequencies by means of two refinement procedures, namely the LASSO iteration and the re-assignment technique. The extraction of the frequencies using wavelet-based methods is described in detail in [4].

The comparison of the method with the results obtained with the FFT envelope methods and time-frequency procedures in terms of accuracy of the detection is given in Table 2.

It is easy to observe that the results obtained with the Goertzel algorithm are overall deviated only with under 1%. We could obtain the same accuracy using the FFT-envelope but this is time-consuming and heuristic as it consists of manual tuning in order to find the integer periods.

6. Conclusion

The paper presented a non-invasive method for the identification of the natural frequencies in cantilever beam. We have combined the extension of the Goertzel to non-integer values and the knowledge about the intervals where the natural frequencies of a known beam could be found in order to obtain a computational efficient algorithm for the identification of these frequencies. We have compared this extraction method with other frequency spectra or time-frequencies extraction procedures in order to show advantages in computational complexity and accuracy in obtaining the harmonics. We suggest to apply the procedures in conjunction with damage detection algorithms like e.g. [1], where such advantages could be exploited.

Regarding the computational complexity, it might be interesting in the future to consider joining the generalized Goertzel algorithm with the technique described in [11]. This way the number of operation could be even lowered in certain configurations.For real-time applications demanding very fast detection times is would be worth to utilize the so-called sliding variant for Goertzel [6], but the accuracy and robustness to noise would have to be evaluated thoroughly.

Should the precision be even increased, one can employ Goertzel in a “zoom fashion”, i.e. after finding peak coarsely, apply Goertzel at a finer-resolution again.

If the speed is the main criterion, one can reduce the number of computations by taking fewer frequency samples and apply some clever peak-picking technique like the one noted in [13] based manipulation with the sinc function.

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Table 1: Comparison of the first ten natural frequencies (in Hz) obtained analytically and with FEM and measurements for the extreme damaged cases. The table shows what is the frequency for the undamaged state obtained analytically from the equation of the beam using the mechanical characteristics and situates this undamaged states within the damaged case simulated via FEM and measured, respectively.

Analytical	FEM		Measurements	
	Low	Up	Low	Up
5.12	4.20	5.30	4.27	5.40
23.55	21.32	25.30	21.27	25.40
62.47	56.7	69.65	54.27	65.40
136.19	133.36	147.34	135.40	148.34
232.45	225.53	257.34	224.40	255.40
342.43	322.56	367.43	320.40	360.40
485.39	471.28	495.01	475.40	495.40
650.25	640.13	656.25	645.40	655.40
826.4	811.33	831.28	815.40	835.40
1035.4	1024.1	1067.82	1020.40	1055.40

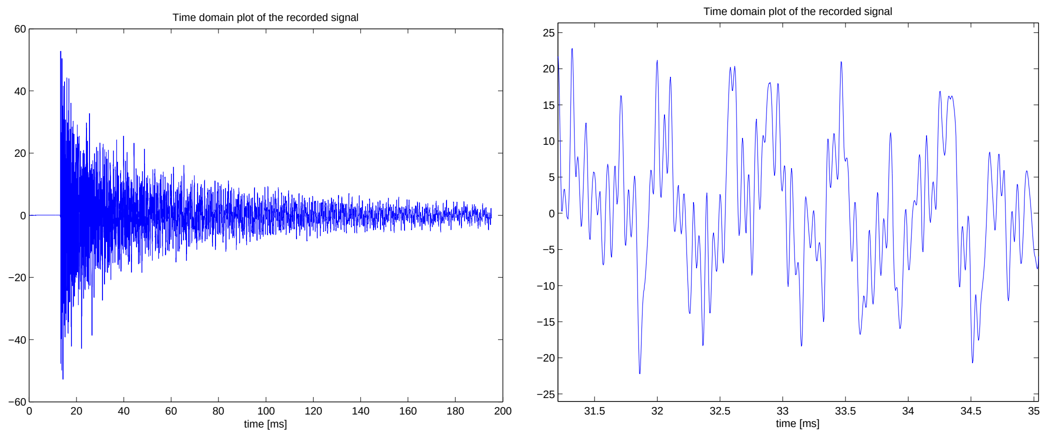

Figure 1: Plot of the cantilever beam signal in the time domain, zoomed at the right.

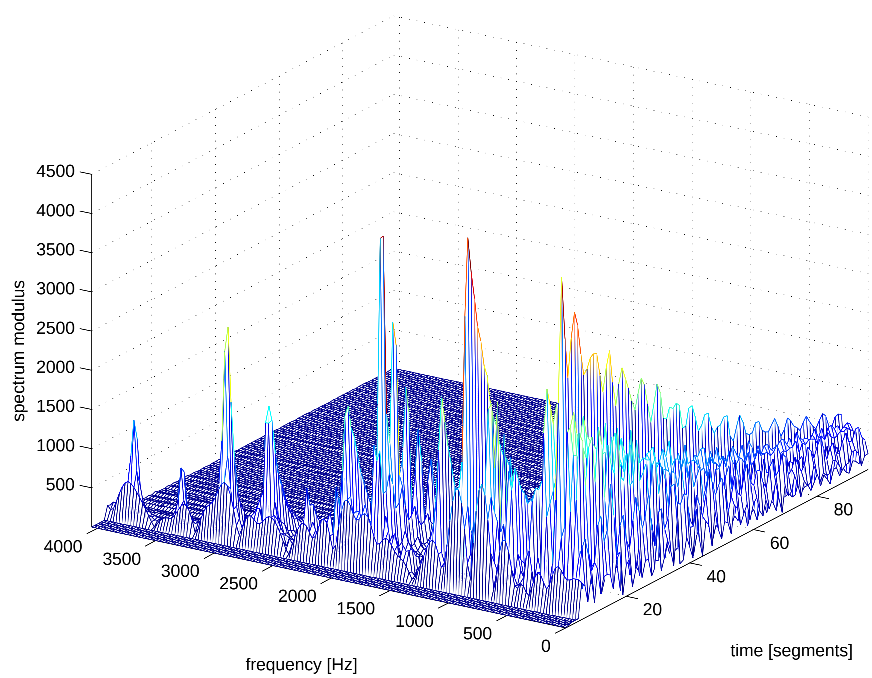

Figure 2: Informative frequency plot over all time segments (FFT modulus). Rectangular window with 1024 samples and half-window overlap was used for this coarse step. Significant contributions of the natural frequencies are visible.

Table 2: Comparison of the first ten natural frequencies obtained using the Goertzel algorithm, the time-frequency and wavelet methods.

Analytical	Goertzel	time-frequency	wavelet-based
5.12	5.12	5.07	4.87
23.55	23.45	22.32	21.32
62.47	62.57	61.7	59.65
136.19	136.09	133.36	137.34
232.45	233.45	235.53	237.34
342.43	342.26	343.56	347.43
485.39	483.39	481.28	485.01
650.25	650.55	645.13	646.25
826.4	827.4	824.33	811.28
1035.4	1034.14	1034.1	1057.82

Inputs: frequency "index" $k \in \mathbb{R}$; signal x of length N

Output: y, representing $X(\omega_k)$ according to eq. (4)

%Precalculation of constants

$A = 2\pi\frac{k}{N}$

$B = 2\cos A$

$C = \mathrm{e}^{-\mathrm{j}A}$

$D = \mathrm{e}^{-\mathrm{j}\frac{2\pi k}{N}(N-1)}$

%State variables

$s_0 = 0$

$s_1 = 0$

$s_2 = 0$

%Main loop

for $i = 0 : N-2$ %one iteration less than traditionally

  $s_0 = x[i] + B \cdot s_1 - s_2$

  $s_2 = s_1$

  $s_1 = s_0$

end

%Finalizing calculations

$s_0 = x[N-1] + B \cdot s_1 - s_2$

$y = s_0 - s_1 \cdot C$

$y = y \cdot D$ %constant substituting the iteration $N-1$, and correcting the phase at the same time

Figure 3: Generalized Goertzel algorithm with shortened iteration loop. The changes, compared to the standard Goertzel algorithm are marked in color.

Inputs: Mechanical characteristics of the beam

Output: Natural frequencies of the beam

I. Load the recorded data

II. Compute the interest intervals using FEM

III. Determine the sampling interval

IV. Determine the window size in samples

V. Split the time samples in segments of interest

VI. Main Goertzel loop over the segments

VII. Determine the maximal value in each interval of interest

VIII. Determine a single such value per interval

Figure 4: Computing the natural frequencies using the generalized Goertzel algorithm

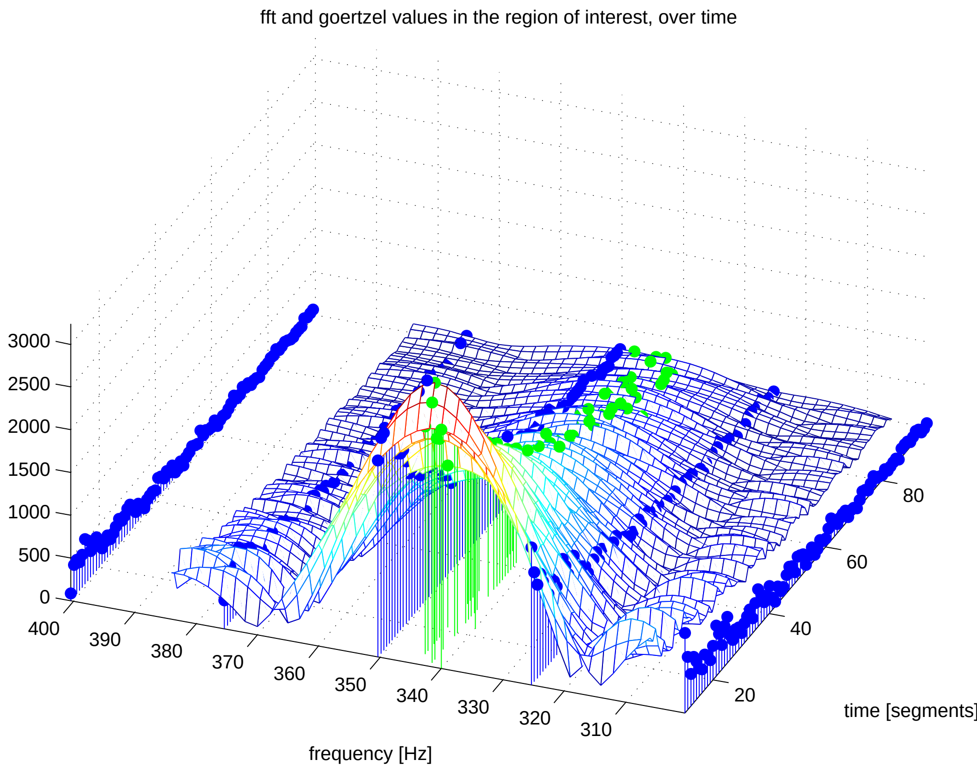

Figure 5: FFT and Goertzel modulus values in the region of interest (the sixth natural frequency). FFT as stems in blue, the mesh of 40 frequencies generated by the Goertzel search, the resultant series of maxima is in green. The separation between FFT bins is 25 Hz = 25600 / 1024.

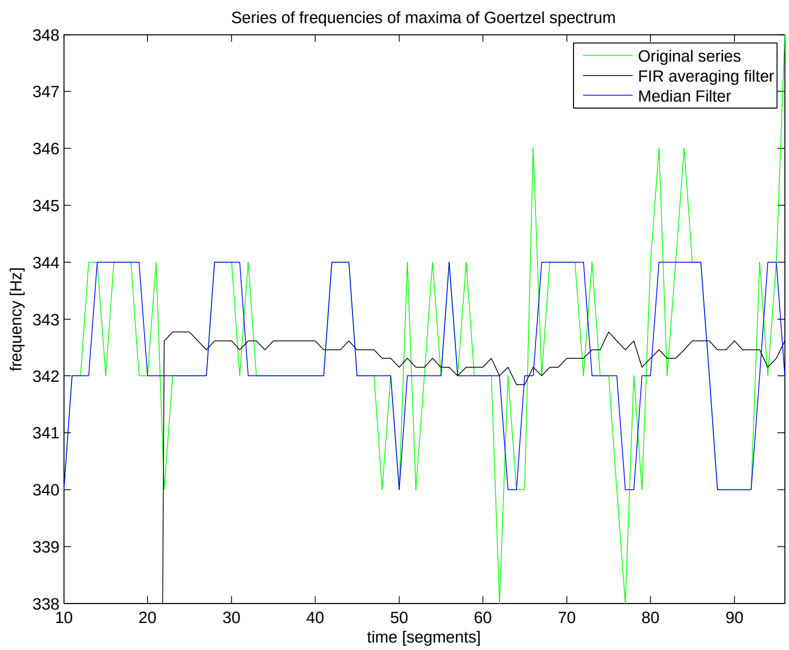

Figure 6: Mean and median filter of the series extracted from Fig. 5 (sixth sinusoid).

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